

The University of Chicago

A CREMONA
TRANSFORMATION GENERATED BY
A PENCIL OF SURFACES

A DISSERTATION SUBMITTED TO THE FACULTY
OF THE DIVISION OF THE PHYSICAL SCIENCES
IN CANDIDACY FOR THE DEGREE OF DOCTOR
OF PHILOSOPHY

DEPARTMENT OF MATHEMATICS
1938

By
FRANK AYRES, JR.

Private Edition, Distributed by
THE UNIVERSITY OF CHICAGO LIBRARIES
CHICAGO, ILLINOIS
1938

The University of Chicago

**A CREMONA
TRANSFORMATION GENERATED BY
A PENCIL OF SURFACES**

A DISSERTATION SUBMITTED TO THE FACULTY
OF THE DIVISION OF THE PHYSICAL SCIENCES
IN CANDIDACY FOR THE DEGREE OF DOCTOR
OF PHILOSOPHY

DEPARTMENT OF MATHEMATICS

1938

By
FRANK AYRES, JR.

Private Edition, Distributed by
THE UNIVERSITY OF CHICAGO LIBRARIES
CHICAGO, ILLINOIS

1938



TABLE OF CONTENTS

	Page
Introduction	1
Section	
1. The transformation	4
2. Plane transformation in planes through d	7
3. Plane transformation in special planes	12
4. Images in the space transformation	12
5. Contact of surfaces along d	14
6. Common lines	15
7. Common curves	18
8. The table of characteristics	20
9. Intersection of the surfaces	21
Vita	25



Digitized by the Internet Archive
in 2026

https://archive.org/details/IA41548403_0050

A CREMONA TRANSFORMATION GENERATED
BY A PENCIL OF SURFACES

Introduction. A Cremona plane transformation, having a characteristic of the form

$$C_1 \sim C_{4k+3} : 4^{2k+1} (2k+1)^2,$$

(where $C_1 \sim C_{4k+3}$ means that the transform of a general line of the plane is a curve of order $4k+3$ and α^β denotes α fundamental points each of multiplicity β) may be defined by means of a correspondence between the conics of a pencil and the points of a suitably chosen line in the plane of the conics. The simplest case may be formulated as follows. In the plane let there be given a pencil of conics and a line d not passing through the base points A_i ($i = 1, 2, 3, 4$) of the pencil. Let the conics of the pencil be related to the points of the line by means of a $(1-1)$ correspondence. A general point R in the plane will determine a unique conic of the pencil and a unique point P on d . If the line PR meets this conic (the associate conic of P) again in R' , the transformation $R \sim R'$ is of the type

$$C_1 \sim C_7 : 4^3 3^2,$$

the fundamental points consisting of the four base points of the pencil each to multiplicity three and three double points which lie on the line d . These latter are the points which lie on their associate conics. The correspondence may be $(1,k)$, k conics of the pencil being associated with a point on d .

The involutory transformation described above is applicable

to spaces of any number of dimensions. In ordinary space the pencil of conics is replaced by a pencil of surfaces of order n having an $(n-2)$ -fold line d and the correspondence relates k surfaces of the pencil to a point on d . E. T. Carroll has studied this transformation when the residual base curve of the pencil is irreducible,¹ and for a reducible base curve when the surfaces are of order three.² Any plane through the $(n-2)$ -fold line d is transformed into itself and the transformation generated in this plane is of the type outlined above, since the plane meets the pencil of surfaces in a pencil of conics. If in one of these planes a conic associated with a point on d is composite, one component passing through the point, this component is a fundamental line of the second kind (a parasitic line), of the transformation. These lines, together with the base curve of the pencil and the $(n-2)$ -fold line d , constitute the fundamental elements of the transformation.

The purpose of the present paper is to study a Cremona transformation, not involutory, which is essentially the product $T \approx T''T'$ of two suitably chosen transformations of the above type. To a point P on the line d we make correspond k surfaces of the pencil, and a point P' on d . A general point R in the space determines a surface of the pencil and an ordered pair of points P and P' on d (cf. Section 1). Let the residual intersection of the line PR and the surface be R' (the transform of R under T'), and the residual intersection of $P'R'$ and the surface be S (the

¹E. T. Carroll, "Systems of involutorial birational transformations contained multiply in special linear line complexes," American Journal of Mathematics, LIV (1932), 707-717.

²E. C. Rusk (= E. T. Carroll), "Cremona involutions defined by a pencil of cubic surfaces," American Journal of Mathematics, LV (1934), 96-108.

transform of R' under T''); then S is the transform of R under T .

In Section 2, the transformation $t \equiv t''t'$ generated in a plane through d is studied. It is of the type

$$C_1 \sim C_{8k+5} : 4^{4k+2} (4k+2)^2,$$

the four $(4k+2)$ -fold fundamental points being the base points of the pencil of conics which is the partial intersection of the plane and the pencil of surfaces. If an associate conic of P passes through that point, then P is a fundamental point of t' and also of t . There are $2k+1$ such points. If an associate conic of P passes through the associate point P' , then P' is a fundamental point of t'' but not of t . The residual intersection of this conic and d is a fundamental point of t . Thus, there are $4k+2$ fundamental points on d and they are double points of the transformation.

The parasitic lines of the transformation T' are fundamental lines of T , while those of T'' are not (cf. Section 6). The residual component of the composite conic of which a parasitic line of T'' is a part is a fundamental line of T . There are on the line d two points which coincide with their associate points. The $k(n-2)$ tangent planes at each point to their k associate surfaces intersect these surfaces in fundamental conics (cf. Section 7). These $2k(n-2)$ conics, together with the fundamental lines, the base curve of the pencil of surfaces, and the $(n-2)$ -fold line d constitute the fundamental elements of the transformation.

The web of surfaces, images of the planes of space, are of order $4k+4n-3$. The line d is $4(n-2)(k+1)$ -fold for each surface and at each point of d the tangent planes to any one of these surfaces coincide in pairs. Moreover, they are common tangent planes

of all the surfaces of the web at the point, and $k(n-2)$ of them coincide with the tangent planes (at the point) of the k surfaces of the pencil associated with that point. A detailed account of the intersection of any two surfaces of the web is given in Section 9.

1. The transformation. In the space of the homogeneous coordinates $(x) \equiv (x_1, x_2, x_3, x_4)$, let

$$F_n(x) \equiv \sum_{i=0}^{n-2} x_1^{n-i-2} x_2^i u_1(x) = 0$$

and

$$F_n'(x) \equiv \sum_{i=0}^{n-2} x_1^{n-i-2} x_2^i v_1(x) = 0$$

where $u_1(x)$ and $v_1(x)$ are quadratic forms in (x) , be two surfaces of degree n each containing an $(n-2)$ -fold line $d \equiv x_1 = x_2 = 0$. The complete intersection of the two surfaces consists, then, of the line d counted $(n-2)^2$ times and a curve $\gamma_{4(n-1)}$, assumed irreducible, of order $4(n-1)$.

Let the pencil of surfaces

$$(1) \quad \lambda_2 F_n(x) - \lambda_1 F_n'(x) = 0$$

be connected with the variable point $(z) \equiv (0, 0, z_3, z_4)$ on the line d by the relation

$$(2) \quad z_3 \varphi_4(\lambda_1, \lambda_2) - z_4 \varphi_3(\lambda_1, \lambda_2) = 0$$

where φ_1 , $(i = 3, 4)$, are binary forms of order k . For each such point (z) , let $(z') \equiv (0, 0, a_3 z_3, a_4 z_4)$, where $a_3 \neq a_4$ and $a_3 a_4 \neq 0$, be its associate point on d .

A general point (y) in the space determines the surface

$$(3) \quad F_n'(y) F_n(x) - F_n(y) F_n'(x) = 0$$

of the pencil and the points

$$(z) \equiv (0, 0, \varphi_3[F_n(y), F_n'(y)], \varphi_4[F_n(y), F_n'(y)])$$

and

$$(z') \equiv (0, 0, a_3 \varphi_3[F_n(y), F_n'(y)], a_4 \varphi_4[F_n(y), F_n'(y)]).$$

A point (x) on the join (yz) , of (y) and (z) has coordinates of the form

$$(4) \quad \varrho x_1 = \tau y_1, \varrho x_2 = \tau y_2, \varrho x_3 = \tau y_3 + \sigma z_3, \varrho x_4 = \tau y_4 + \sigma z_4,$$

and the point (x) , the residual intersection of (yz) and (3) is determined by applying the equations (4) to (3). The result in its simplified form is

$$\begin{aligned} & \sigma \{F_n'(y) \sum_{i=0}^{n-2} \frac{y_1^{n-1-2i}}{y_2} u_1(\varphi) - F_n(y) \sum_{i=0}^{n-2} \frac{y_1^{n-1-2i}}{y_2} v_1(\varphi)\} \\ & + 2\tau \{F_n'(y) \sum_{i=0}^{n-2} \frac{y_1^{n-1-2i}}{y_2} u_1(y, \varphi) - F_n(y) \sum_{i=0}^{n-2} \frac{y_1^{n-1-2i}}{y_2} v_1(y, \varphi)\} = 0, \end{aligned}$$

in which $u_1(\varphi)$ and $v_1(\varphi)$ are quadratic forms in

$$(z) \equiv (0, 0, \varphi_3[F_n(y), F_n'(y)], \varphi_4[F_n(y), F_n'(y)])$$

and $u_1(y, \varphi)$ and $v_1(y, \varphi)$ are the polar forms of (z) with respect to $u_1(y)$ and $v_1(y)$, respectively. Hence,

$$\tau \equiv F_n'(y) \sum_{i=0}^{n-2} \frac{y_1^{n-1-2i}}{y_2} u_1(\varphi) - F_n(y) \sum_{i=0}^{n-2} \frac{y_1^{n-1-2i}}{y_2} v_1(\varphi),$$

and

$$\sigma \equiv -2 \left\{ F_n'(y) \sum_{i=0}^{n-2} \frac{y_1^{n-1-2i}}{y_2} u_1(y, \varphi) - F_n(y) \sum_{i=0}^{n-2} \frac{y_1^{n-1-2i}}{y_2} v_1(y, \varphi) \right\}.$$

A point (x) on the join (Xz') , of (X) and (z') has coordinates of the form

$$(5) \quad \varrho' x_1 = \tau' X_1, \quad \varrho' x_2 = \tau' X_2, \quad \varrho' x_3 = \tau' X_3 + \sigma' Z_3', \quad \varrho' x_4 = \tau' X_4 + \sigma' Z_4'$$

and the point (y') , the residual intersection of (Xz') and (3), is determined as before. After making the replacements for X_1 as given by (4), the coordinates of (y') have the form

$$(5') \quad \theta y_1' = \tau \tau' y_1, \quad \theta y_2' = \tau \tau' y_2, \quad \theta y_3' = \tau \tau' y_3 + (\tau \sigma' + a_3 \sigma') \varphi_3(F, F'),$$

$$\theta y_4' = \tau \tau' y_4 + (\tau \sigma' + a_4 \sigma') \varphi_4(F, F'),$$

with

$$\tau' = F_n'(y) \sum_{i=0}^{n-2} y_1^{n-1-2i} y_2^i u_1(a\varphi) - F_n(y) \sum_{i=0}^{n-2} y_1^{n-1-2i} y_2^i v_1(a\varphi)$$

and

$$\begin{aligned} \sigma' &= -2 \left\{ \tau [F_n'(y) \sum_{i=0}^{n-2} y_1^{n-1-2i} y_2^i u_1(y, a\varphi) - F_n(y) \sum_{i=0}^{n-2} y_1^{n-1-2i} y_2^i v_1(y, a\varphi)] \right. \\ &\quad \left. + \sigma [F_n'(y) \sum_{i=0}^{n-2} y_1^{n-1-2i} y_2^i u_1(\varphi, a\varphi) - F_n(y) \sum_{i=0}^{n-2} y_1^{n-1-2i} y_2^i v_1(\varphi, a\varphi)] \right\} \\ &\equiv -2 \{ \tau \xi + \sigma \eta \}. \end{aligned}$$

The transformation is of order $4nk+4n-3$, since the order of τ and of τ' is $2nk+2n-2$, of σ is $nk+2n-1$, and of σ' is $3nk+4n-3$. The line d appears to multiplicity $2(n-2)(k+1)$ on the surfaces $\tau_2(nk+n-1) = 0$ and $\tau_2'(nk+n-1) = 0$, to multiplicity $(n-2)(k+2)$ on $\sigma_{nk+2n-1} = 0$, to multiplicity $(n-2)(3k+4)$ on $\sigma_{3nk+4n-3}' = 0$, and to multiplicity $4(n-2)(k+1)$ on $S_{4nk+4n-3} = 0$, the image of a general plane. The base curve $\gamma_{4(n-1)}$ appears to multiplicity $2k+1$ on each of the surfaces $\tau_2(nk+n-1) = 0$ and $\tau_2'(nk+n-1) = 0$, to multiplicity $k+1$ on $\sigma_{nk+2n-1} = 0$, to multiplicity $3k+2$ on $\sigma_{3nk+4n-3}' = 0$ and to multiplicity $4k+2$ on $S_{4nk+4n-3} = 0$.

2. Plane transformation in planes through d. A plane through d intersects the pencil of surfaces (1) in a pencil of conics. The base points of this pencil, A_i ($i = 1, 2, 3, 4$), are the four points of intersection of the plane and $\gamma_{4(n-1)}$ which are not on d.

Every plane through d is transformed by (5') into itself. In one of these planes the pencil of conics may be represented by the equation

$$(6) \quad \lambda_2 C_1(x) - \lambda_1 C_2(x) = 0,$$

where $C_1(x) \equiv \sum_1^3 a_{ij} x_i x_j$ and $C_2(x) \equiv \sum_1^3 b_{ij} x_i x_j$, and the line d may be taken to have the equation $x_3 = 0$. Let the conics of the pencil be connected with a variable point $(z) \equiv (z_1, z_2, 0)$ on d by the relation

$$(7) \quad z_1 \varphi_2(\lambda_1, \lambda_2) - z_2 \varphi_1(\lambda_1, \lambda_2) = 0,$$

where $\varphi_i(\lambda)$, ($i = 1, 2$), are binary forms of order k. For each such point (z) , let $(z') \equiv (a_1 z_1, a_2 z_2, 0)$, where $a_1 \neq a_2$ and $a_1 a_2 \neq 0$, be its associate point.

A general point (y) in the plane determines the conic of the pencil through it,

$$(8) \quad C_2(y) C_1(x) - C_1(y) C_2(x) = 0$$

and the points

$$(z) \equiv (\varphi_1[C_1(y), C_2(y)], \varphi_2[C_1(y), C_2(y)], 0)$$

and

$$(z') \equiv (a_1 \varphi_1[C_1(y), C_2(y)], a_2 \varphi_2[C_1(y), C_2(y)], 0).$$

A point (x) on the join (yz) of (y) and (z) has coordi-

nates of the form

$$(9) \quad \varrho x_1 = \tau y_1 + \sigma z_1, \varrho x_2 = \tau y_2 + \sigma z_2, \varrho x_3 = \tau y_3,$$

and the point (X) , the residual intersection of (yz) and (8) is determined by applying the equations (9) to (8). The result in its simplified form is

$$\sigma \{C_2(y)C_1(\varphi) - C_1(y)C_2(\varphi)\} + 2\tau \{C_2(y)C_1(y, \varphi) - C_1(y)C_2(y, \varphi)\} = 0,$$

in which $C_i(\varphi)$, $(i = 1, 2)$, are quadratic forms in $(\varphi_1, \varphi_2, 0)$ and $C_i(y, \varphi)$ are the polar forms of $(\varphi_1, \varphi_2, 0)$ with respect to $C_i(y)$. Hence,

$$\tau = C_2(y) C_1(\varphi) - C_1(y) C_2(\varphi)$$

and

$$\sigma = -2 \{C_2(y) C_1(y, \varphi) - C_1(y) C_2(y, \varphi)\}.$$

A point (x) on the join (Xz') of (X) and (z') has coordinates of the form

$$(10) \quad \varrho' x_1 = \tau' X_1 + \sigma' z'_1, \varrho' x_2 = \tau' X_2 + \sigma' z'_2, \varrho' x_3 = \tau' X_3,$$

and the point (y') , the residual intersection of (Xz') and (8) is determined as before. After making the replacements for X_i as given by (9), the coordinates of (y) have the form

$$(10') \quad \theta y'_1 = \tau \tau' y_1 + (\tau' \sigma + a_1 \sigma') \varphi_1, \theta y'_2 = \tau \tau' y_2 + (\tau' \sigma + a_2 \sigma') \varphi_2,$$

$$\theta y'_3 = \tau \tau' y_3,$$

with

$$\tau' = C_2(y) C_1(a\varphi) - C_1(y) C_2(a\varphi),$$

and

$$\sigma' = -2 \{ \tau [C_2(y) C_1(y, a\varphi) - C_1(y) C_2(y, a\varphi)] + \sigma [C_2(y) C_1(\varphi, a\varphi) - C_1(y) C_2(\varphi, a\varphi)] \} = -2 \{ \tau \} + \sigma \eta \}.$$

The transformation is of order $8k+5$, since τ and τ' are of order $4k+2$, σ of order $2k+3$, and σ' of order $6k+5$. The base points A_1 occur to multiplicity $2k+1$ on the curves $\tau = 0$ and $\tau' = 0$, to multiplicity $k+1$ on $\sigma = 0$, $3k+2$ on $\sigma' = 0$, and $4k+2$ on the image curve C_{8k+5} of a general line of the plane.

If α_i represents the number of fundamental points of multiplicity i , then, since $\alpha_{4k+2} = 4$, the fundamental equations

$$\sum \alpha_i \cdot i = 24k+12, \quad \sum \alpha_i \cdot i^2 = 64k^2 + 80k + 24,$$

require that $\alpha_2 = 4k+2$ and $\alpha_i = 0$ for all other values of i .

Hence, the fundamental points of the transformation are as follows, the superscript denoting the multiplicity,

$$A_1^{4k+2} A_2^{4k+2} A_3^{4k+2} A_4^{4k+2} P_1^2 P_2^2 \dots P_{2k+1}^2 Q_1^2 Q_2^2 \dots Q_{2k+1}^2$$

The fundamental points, aside from the base points A_1 , are the points common to $\tau = 0$ and $\sigma = 0$ (denoted by P_1), and those common to $\tau' = 0$ and $\sigma' = 0$ (denoted by Q_1). The conic of the pencil determined by R , a general point of $\tau = 0$, meets $\tau = 0$ in the base points A_1 , each counted $2k+1$ times, and the point R . Hence, $\tau = 0$ is composite, consisting of $2k+1$ conics of the pencil. Each of these conics meets $\sigma = 0$ in $4k+6$ points of which the base points A_1 account for $4k+4$. Since $\sigma = 0$ is the locus of the invariant points of the transformation (9), $\tau = 0$ and $\sigma = 0$ are tangent at each of the points P_1 . Similarly, $\tau' = 0$ consists of $2k+1$ conics of the pencil and each of these conics is tangent to $\sigma' = 0$ at one of the points Q_1 . Each point P_1 is a point (z) on d such that one of its associate conics passes through it, and the conic in question is a fundamental conic. If one of the associate conics of a point (z) passes through the associate point (z') , then the

further intersection of this conic and d is one of the points Q_1 . Thus, there are $4k+2$ fundamental conics belonging to the pencil, each of which passes through one of the fundamental points and is the image of that point. There are four fundamental curves, $\alpha_{4k+2,1}$, each of order $4k+2$, images of the base points A_1 . The image of the point, say, A_1 passes $2k$ times through A_1 , $2k+1$ times through each of the other base points, and simply through the $4k+2$ fundamental points on d .

The invariant elements of the transformation consist of the line d , the $2k$ conics of the pencil associates of the points $(1,0,0)$ and $(0,1,0)$, and the double points D_1 of the three degenerate conics of the pencil.

The image of any point on $\tau = 0$ is a definite point on d ; likewise, the image of a point on $\tau' = 0$ is a definite point on d .

The curve $\sigma = 0$ is the locus of points invariant under the transformation (9); the curve $\sigma' = 0$ is the locus of points whose transform under (10') is the same as that under (9). For simplicity, they shall be called invariant under (10).

The line d intersects $\sigma = 0$ in the points P_1 and the points T_1 and T_2 , the points of tangency of the two conics of the pencil which are tangent to d ; $\sigma' = 0$ in the points P_1 , counted twice since they are double points for that curve, the points Q_1 , and T_1 and T_2 ; and C_{8k+5} , the transform of a general line, in the points P_1 and Q_1 each counted twice, and the point of intersection of the general line and d .

Except for the cases noted below, the intersection of any two of the curves associated with the transformation are readily accounted for by the fundamental points. The curve $\sigma = 0$ intersects each of the conics of $\tau' = 0$ in the points A_1 counted $k+1$ times

and in the two points of tangency of the tangents to the conic drawn from its associate point (z); $\sigma' = 0$ in the points A_1 counted $(k+1)(3k+2)$ times, the points T_1 and D_1 , the $4k$ points of tangency of the tangents drawn from the points $(1,0,0)$ and $(0,1,0)$ to their respective associate conics, and the points P_1 counted twice; C_{8k+5} in the points A_1 counted $(k+1)(4k+2)$ times, the points P_1 counted twice, and $6k+5$ points invariant under (9); and $\alpha_{4k+2,j}$ in the point A_j counted $2k(k+1)$ times, the points A_k counted $(2k+1)(k+1)$ times, the points P_1 , and $3k+2$ points invariant under (9).

The curve $\sigma' = 0$ intersects C_{8k+5} in the points A_1 counted $(4k+2)(3k+2)$ times, the points P_1 and Q_1 counted four and two times respectively, and $2k+3$ points invariant under (10); and $\alpha_{4k+2,j}$ in the point A_j counted $2k(3k+2)$ times, the points A_k counted $(2k+1)(3k+2)$ times, the points P_1 counted twice, the points Q_1 , and $k+1$ points invariant under (10).

The image curves of two general lines of the plane intersect in the points A_1 counted $(4k+2)^2$ times, the points P_1 and Q_1 each counted four times, and the point of intersection of the two lines.

The table of characteristics may be expressed in the following form where $C_1 \sim C_{8k+5}$ means that the transform of the line C_1 is the curve C_{8k+5} , $\pi_{2,1}$ and $\varrho_{2,1}$ represent fundamental conics, and J_{24k+12} is the Jacobian of the net of curves $|C_{8k+5}|$.

$$C_1 \sim C_{8k+5} : A_1^{4k+2} A_2^{4k+2} A_3^{4k+2} A_4^{4k+2} P_1^2 \dots P_{2k+1}^2 Q_1^2 \dots Q_{2k+1}^2$$

$$P_1 \sim \pi_{2,1} : A_1 A_2 A_3 A_4 P_1$$

$$Q_1 \sim \varrho_{2,1} : A_1 A_2 A_3 A_4 Q_1$$

$$A_1 \sim \alpha_{4k+2,1} : A_1^{2k} A_j^{2k+1} A_k^{2k+1} A_m^{2k+1} P_1 P_2 \dots P_{2k+1} Q_1 \dots Q_{2k+1}$$

$$\tau_{4k+2} : (2k+1)\pi_{2,1}$$

$$\tau'_{4k+2} : (2k+1)\varrho_{2,1}$$

$$\sigma_{2k+3} : A_1^{k+1} A_2^{k+1} A_3^{k+1} A_4^{k+1} P_1 P_2 \dots P_{2k+1}$$

$$\sigma'_{6k+5} : A_1^{3k+2} A_2^{3k+2} A_3^{3k+2} A_4^{3k+2} P_1^2 \dots P_{2k+1}^2 Q_1 \dots Q_{2k+1}$$

$$J_{24k+12} : \alpha_{4k+2,1} + \alpha_{4k+2,2} + \alpha_{4k+2,3} + \alpha_{4k+2,4} \\ + (2k+1)\pi_{2,1} + (2k+1)\varrho_{2,1}$$

3. Plane transformations in special planes. In general, the curves $\tau = 0$ and $\tau' = 0$ have only the base points A_1 in common. If, however, the point $(1,0,0)$ or $(0,1,0)$ is a fundamental point (z) for the plane, the image conic of the point is a component of both $\tau = 0$ and $\tau' = 0$. It is also a component of $\eta = 0$ and $\sigma' = 0$.

If a fundamental conic, image of a point (z) on d , passes also through the associate point (z') , this conic is a part of the curves $\tau = 0$ and $\tau' = 0$. If a fundamental conic, image of a point (z) on d ,

$$\lambda_2 C_1(x) - \lambda_1 C_2(x) = 0$$

is a component of $\eta = 0$, then

$$\lambda_2 C_1(\varphi, x) - \lambda_1 C_2(\varphi, x) = 0$$

for all point $(x) \equiv (x_1, x_2, 0)$. Hence, if a fundamental conic is tangent to the line d at its associate point, the conic is a component of $\tau = 0$, $\eta = 0$ and $\sigma' = 0$.

4. Images in the space transformation. The image of the line d is the surface $\tau_{2(nk+n-1)} \cdot \tau'_{2(nk+n-1)} = 0$. The surface

$\sigma_{nk+2n-1} = 0$ is the locus of points invariant under the transformation (4). When the transform of a point under (5') coincides with the transform of the point under (4), the point lies on the surface $\sigma'_{3nk+4n-3} = 0$. For the sake of brevity, they shall be called invariant under (5). The invariant elements under (5') will be found in Section 7.

The image of a point (z) on d is a curve of order $4(n-2)(k+1)$ consisting of (a) $k(n-2)$ conics in the $k(n-2)$ planes through d in which (z) is a fundamental point P_1 ; (b) $k(n-2)$ conics in the $k(n-2)$ planes through d in which (z) is a fundamental point Q_1 (cf. Section 8); and (c) the line counted $4(n-2)$ times. The conics of the first set lie on $\tau_{2(nk+n-1)} = 0$ while those of the second set lie on $\tau'_{2(nk+n-1)} = 0$.

The image of an arbitrary line L is a curve of order $4nk+4n-3$. If L meets d, the proper image is a curve of order $8k+5$ since the image of the intersection of L and d is a curve of order $4(n-2)(k+1)$. This curve, C_{8k+5} , lies in the plane of L and d and has been studied in Section 2.

The image of a general plane $S_1 \equiv (bx) \equiv \sum_{i=1}^4 b_i x_i = 0$ is the surface

$$S_{4nk+4n-3} \equiv \tau\tau'(bx) + (\tau'\sigma + a_3\sigma')b_3\varphi_3 + (\tau'\sigma + a_4\sigma')b_4\varphi_4 = 0.$$

The image of a point on $\gamma_{4(n-1)}$ is the plane curve of order $4k+2$ discussed in Section 2.

The image of a surface, $F = 0$, of the pencil (1) is a surface of order $n(4nk+4n-3)$ and is composed of (a) the given surface; (b) the surfaces $\tau_{2nk+2n-2} = 0$ and $\tau'_{2nk+2n-2} = 0$ each to multiplicity $n-2$; and (c) a surface $\tau_{8(nk+n-1)} = 0$, the image of $\gamma_{4(n-1)}$. This latter surface contains d to multiplicity $8(n-2)(k+1)$ and

$\gamma_{4(n-1)}$ to multiplicity $8k+3$.

5. Contact of surfaces along d. For simplicity, the details of the analysis are given below for the case $n = 3$, $k = 1$. Consider the cubic surface

$$z_4(x_1u(x) + x_2v(x)) - z_3(x_1w(x) + x_2t(x)) = 0,$$

where $u(x)$, $v(x)$, $w(x)$, and $t(x)$ are quadratic forms in (x_1, x_2, x_3, x_4) , whose associate point on d is $(z) \equiv (0, 0, z_3, z_4)$. The tangent plane to this surface at (z) is given by

$$(z_4\bar{F} - z_3\bar{F}') \equiv z_4(x_1u(z) + x_2v(z)) - z_3(x_1w(z) + x_2t(z)) = 0.$$

The degrees of τ , τ' , σ , σ' , S , and Γ in this case are equal respectively to 10, 10, 8, 18, 21, and 40. Denoting by τ_0 , τ'_0 , σ_0 , σ'_0 , S_0 , and Γ_0 the configuration of tangent planes to $\tau_{10} = 0$, $\tau'_{10} = 0$, $\sigma_8 = 0$, $\sigma'_{18} = 0$, $S_{21} = 0$, and $\Gamma_{40} = 0$ respectively, at the point (z) , we have

$$\tau_0 \equiv (z_4\bar{F} - z_3\bar{F}')\{z_3(2A\bar{F} + B\bar{F}') + z_4(B\bar{F} + 2C\bar{F}')\} = 0,$$

$$\tau'_0 \equiv (a_3z_4\bar{F} - a_4z_3\bar{F}')\{z_3(2a_3A\bar{F} + a_4B\bar{F}') + z_4(a_3B\bar{F} + 2a_4C\bar{F}')\} = 0,$$

$$\sigma_0 \equiv -2(z_4\bar{F} - z_3\bar{F}')\{Az_3^2 + Bz_3z_4 + Cz_4^2\} = 0,$$

$$\sigma'_0 \equiv -2(z_4\bar{F} - z_3\bar{F}')^2\{Az_3^2 + Bz_3z_4 + Cz_4^2\}$$

$$\{z_3(2a_3A\bar{F} + a_4B\bar{F}') + z_4(a_3B\bar{F} + 2a_4C\bar{F}')\} = 0,$$

$$S_0 \equiv (z_4\bar{F} - z_3\bar{F}')^2\{z_3(2a_3A\bar{F} + a_4B\bar{F}') + z_4(a_3B\bar{F} + 2a_4C\bar{F}')\}^2 = 0,$$

and

$$\Gamma_0 \equiv (z_4\bar{F} - z_3\bar{F}')^4\{z_3(2a_3A\bar{F} + a_4B\bar{F}') + z_4(a_3B\bar{F} + 2a_4C\bar{F}')\}^4 = 0,$$

where

$$\begin{aligned}
 A &= (x_1 w_{34} + x_2 t_{34})(x_1 u_{33} + x_2 v_{33}) \\
 &\quad - (x_1 u_{34} + x_2 v_{34})(x_1 w_{33} + x_2 t_{33}), \\
 B &= (x_1 w_{44} + x_2 t_{44})(x_1 u_{33} + x_2 v_{33}) \\
 &\quad - (x_1 u_{44} + x_2 v_{44})(x_1 w_{33} + x_2 t_{33}), \\
 C &= (x_1 w_{44} + x_2 t_{44})(x_1 u_{34} + x_2 v_{34}) \\
 &\quad - (x_1 u_{44} + x_2 v_{44})(x_1 w_{34} + x_2 t_{34}).
 \end{aligned}$$

In the general case, $k(n-2)$ of the tangent planes to $\tau_{2(nk+n-1)} = 0$, at any point (z) on the $(n-2)$ -fold line d , coincide with the tangent planes to the k surfaces of the pencil (1), associates of (z) . They are found simply among those of $\sigma_{nk+2n-1} = 0$, doubly among those of $\sigma'_{3nk+4n-3} = 0$ and $S_{4nk+4n-3} = 0$, and four-fold among those of $\Gamma_{8(nk+n-1)} = 0$ at the point. The surfaces $\tau'_{2(nk+n-1)} = 0$ and $\sigma'_{3nk+4n-3} = 0$ have $(k+2)(n-2)$ tangent planes at any point on d in common, which are found doubly among those of $S_{4nk+4n-3} = 0$ and four-fold among those of $\Gamma_{8(nk+n-1)} = 0$. The surfaces $\sigma_{nk+2n-1} = 0$ and $\sigma'_{3nk+4n-3} = 0$ have $2(n-2)$ other common tangent planes (at each point of d) which are also among those of the surface $\eta_{2(nk+n-1)} = 0$ at the point.

Any two surfaces of the web $|S_{4nk+4n-3}|$, images of the planes of space, have at each point on d all their tangent planes in common. The ∞^2 surfaces of the web, images of the bundle of planes passing through a point on d have in common the $k(n-2)$ fundamental conics which lie in the $k(n-2)$ tangent planes common to the k surfaces of the pencil, associates of the point.

6. Common lines. There are $6nk+6n-8k-10$ fundamental

lines (g) occurring simply on $\tau_{2(nk+n-1)} = 0$ and $\sigma_{nk+2n-1} = 0$.¹ For the transformation (4) they are parasitic and their number may be determined as follows:

The configuration of the tangent planes to a surface

$$\lambda_4 \sum_{i=0}^{n-2} x_1^{n-1-2i} x_2^i u_1(x) - \lambda_3 \sum_{i=0}^{n-2} x_1^{n-1-2i} x_2^i v_1(x) = 0$$

of the pencil (1) at its associate point $(z) \equiv (0, 0, \varphi_3, \varphi_4)$ is given by

$$(11) \quad \lambda_4 \sum_{i=0}^{n-2} x_1^{n-1-2i} x_2^i u_1(\varphi) - \lambda_3 \sum_{i=0}^{n-2} x_1^{n-1-2i} x_2^i v_1(\varphi) = 0.$$

Let the equation of one of these planes have the form $x_1 = rx_2$. When this is combined with the equation of the surface the result is

$$(12) \quad \lambda_4 \sum_{i=0}^{n-2} r^{n-1-2i} u_1(x) - \lambda_3 \sum_{i=0}^{n-2} r^{n-1-2i} v_1(x) = 0.$$

Let the equation of the conic (12) be represented by

$$(12') \quad K_2 \equiv \sum_2^4 c_{ij} x_i x_j = 0,$$

where the coefficient c_{22} is of order n in r and one in λ ; c_{23} and c_{24} of order $n-1$ in r and one in λ ; and c_{33} , c_{34} , and c_{44} are of order $n-2$ in r and one in λ . The condition that (12') shall represent a pair of straight lines is that the discriminant $\Delta \equiv |c_{ij}|$ shall vanish. When λ is eliminated between the equation of the tangent planes (11),

$$(11') \quad \lambda_4 \sum_{i=0}^{n-2} r^{n-1-2i} u_1(\varphi) - \lambda_3 \sum_{i=0}^{n-2} r^{n-1-2i} v_1(\varphi) \equiv T_0 \lambda^{2k+1} + T_1 \lambda^{2k} + \dots + T_{2k+1} = 0,$$

¹Carroll, "Systems of involutorial birational transformations," op. cit., LIV, 714.

and

$$\Delta = B_0 \lambda^3 + B_1 \lambda^2 + B_2 \lambda + B_3 = 0,$$

the resultant is of the form $\sum B_i^{2k+1} \cdot T_j^3$, where B_1 is of order $3n-4$ in r and T_j is of order $n-2$ in r . There are, then $3(n-2) + (2k+1)(3n-4)$ such lines.

The plane determined by one of the lines g and the $(n-2)$ -fold line d is tangent to the surface of the pencil at its associate point (z) , the intersection of g and d . The residual intersection of the plane and surface is a composite conic consisting of the line g and a line h . The lines (g) occur simply on $\sigma'_{3nk+4n-3} = 0$ and $S_{4nk+4n-3} = 0$, and doubly on $\Gamma_{8(nk+n-1)} = 0$.

Corresponding to the surfaces $\tau_{2(nk+n-1)} = 0$ and $\sigma_{nk+2n-1} = 0$ obtained by means of transformation (4), we obtain in a similar manner the surfaces $\mu = 0$ and $v = 0$ when the points (y) of a surface, associate of (z) , are joined to the points (z') . When the results of this transformation are compared with those of Section 1, it is found that $\mu = \tau'$ and $v = -2\xi$. There are $6nk+6n-8k-10$ fundamental lines (g') occurring simply on $\mu = 0$ and $v = 0$ but not on $\sigma'_{3nk+4n-3} = 0$. In the plane of one of these lines, g' , and the $(n-2)$ -fold line d , the component h' of the conic of intersection of the plane and the surface determined by g' meets d on one of the fundamental points Q_1 of the plane. The lines h' occur simply on $\sigma'_{3nk+4n-3} = 0$, $\tau'_{2(nk+n-1)} = 0$, and $S_{4nk+4n-3} = 0$ and doubly on $\Gamma_{8(nk+n-1)} = 0$.

Under the transformation (5'), the transform of any point of the line g is the line h while the transform of any point on h' is the line g' . Thus, neither the lines (g) nor (h') are parasitic lines in the ordinary meaning of the term.

7. Common curves. At the points $(z) \equiv (0,0,0,1)$ and $(0,0,1,0)$, $k(n-2)$ of the tangent planes to $\tau_{2(nk+n-1)} = 0$ coincide with $k(n-2)$ of the respective tangent planes to $\tau'_{2(nk+n-1)} = 0$. Thus $\tau_{2(nk+n-1)} = 0$ and $\tau'_{2(nk+n-1)} = 0$ have the $2k(n-2)$ conics which lie in these planes and pass through the point of tangency in common. They are single on each of $\tau_{2(nk+n-1)} = 0$, $\tau'_{2(nk+n-1)} = 0$, and $\sigma'_{3nk+4n-3} = 0$; are double on $S_{4nk+4n-3} = 0$; and four-fold on $\Gamma_{8(nk+n-1)} = 0$. The surfaces $\tau'_{2(nk+n-1)} = 0$ and $\sigma'_{3nk+4n-3} = 0$ are tangent along each of these conics. They are fundamental conics of the transformation and will be denoted by $I_{4k(n-2)}$.

The configuration of the tangent planes to the surface

$$\lambda_4 \sum_{i=0}^{n-2} x_1^{n-i-1} x_2^i u_1(x) - \lambda_3 \sum_{i=0}^{n-2} x_1^{n-i-1} x_2^i v_1(x) = 0$$

of Section 6 at the point $(z') \equiv (0,0,a_3\varphi_3,a_4\varphi_4)$ is given by

$$\lambda_4 \sum_{i=0}^{n-2} x_1^{n-i-1} x_2^i u_1(a\varphi) - \lambda_3 \sum_{i=0}^{n-2} x_1^{n-i-1} x_2^i v_1(a\varphi) = 0.$$

When the replacement $x_1 = rx_2$ is made in this equation and that of (11) and λ is eliminated between them, the resultant is of order $(4k+2)(n-2)$ in r . The residual intersection of one of these planes with the associate surface is a conic which passes through the associate points (z) and (z') . Of these conics, $2k(n-2)$ constitute the set $I_{4k(n-2)}$, corresponding to the case when the points (z) and (z') coincide. The remaining $2(k+1)(n-2)$ conics are common to the surfaces $\tau_{2(nk+n-1)} = 0$ and $\tau'_{2(nk+n-1)} = 0$.

The conic (12') is a fundamental conic of the plane $x_1 = rx_2$ and, hence, is a component of the curve $\tau = 0$, the intersection of the plane and $\tau_{2(nk+n-1)} = 0$. The condition that it also be a component of the curve $\eta = 0$, the intersection of the plane and the surface $\eta_{2(nk+n-1)} = 0$, is $c_{34}\varphi_3 + c_{44}\varphi_4 = 0$.

This condition is of order $n-2$ in r and $2k+1$ in λ and when λ is eliminated between it and (11'), the resultant is of order $(4k+2)(n-2)$ in r . These conics are common to $\tau_{2(nk+n-1)} = 0$ and $\eta_{2(nk+n-1)} = 0$ and, hence, lie on $\sigma'_{3nk+4n-3} = 0$. They consist, in part, of the $2k(n-2)$ conics, $I_{4k(n-2)}$; the remaining $2(k+1)(n-2)$ conics being those fundamental conics (of their planes) which are tangent to the line d .

The first polar surface of a point (z) on d with respect to one of its associate surfaces is of order $n-1$ and contains d to multiplicity $n-2$. The two surfaces intersect in the line d to multiplicity $(n-2)^2$ and a curve of order $3n-4$. When (z) is taken as the point $(0,0,0,1)$ or $(0,0,1,0)$, the k curves of order $3n-4$ determined by each point are common to both $\sigma_{nk+2n-1} = 0$ and $\sigma'_{3nk+4n-3} = 0$. The $2k$ curves will be denoted by $C_{2k(3n-4)}$.

The complete intersection of the surfaces $\sigma_{nk+2n-1} = 0$ and $\xi_{nk+2n-1} = 0$, the invariant surface of the transformation described in Section 6, consists of the line d to multiplicity $(n-2)^2(k+2)^2$; $\gamma_{4(n-1)}$ to multiplicity $(k-1)^2$; the line d counted $4(n-2)$ times more due to contact along d ; $C_{2k(3n-4)}$; and a curve, C_{4n-3} . This latter curve is the locus of points such that the tangent plane at the point to the surface of the pencil determined by it passes through d .

The invariant elements of the transformation (5') are, thus, the $4k$ surfaces of the pencil, associates of the points $(0,0,0,1)$ and $(0,0,1,0)$, and the curve C_{4n-3} .

In the case of the pairs of surfaces $\tau'_{2(nk+n-1)} = 0$ and $\sigma_{nk+2n-1} = 0$; $\sigma_{nk+2n-1} = 0$ and $S_{4nk+4n-3} = 0$; $\sigma_{nk+2n-1} = 0$ and $\Gamma_{8(nk+n-1)} = 0$; $\sigma'_{3nk+4n-3} = 0$ and $S_{4nk+4n-3} = 0$; and $\sigma'_{3nk+4n-3} = 0$ and $\Gamma_{8(nk+n-1)} = 0$; there are common space curves

as is seen from Section 2, which gives the number of points of intersection of the curves of intersection of these surfaces with a plane through d . Their orders will be given in Section 9.

8. The table of characteristics. The general table of characteristics has the form:

$$\begin{aligned}
 d &\sim \left\{ \begin{array}{l} \tau_{2(nk+n-1)} : d^{2(n-2)(k+1)}, k(n-2)\bar{d}, \gamma_{4(n-1)}^{2k+1}, \\ [6nk+6n-8k-10]g, I_{4k(n-2)}, \\ \tau'_{2(nk+n-1)} : d^{2(n-2)(k+1)}, (n-2)(k+2)\bar{\bar{d}}, \gamma_{4(n-1)}^{2k+1}, \\ [6nk+6n-8k-10]h', I_{4k(n-2)}, \end{array} \right. \\
 S_1 &\sim S_{4nk+4n-3} : d^{4(n-2)(k+1)}, k(n-2)\bar{d}^2, (n-2)(k+2)\bar{\bar{d}}^2, \gamma_{4(n-1)}^{4k+2}, \\ &\quad [6nk+6n-8k-10]g, [6nk+6n-8k-10]h', I_{4k(n-2)}^2, \\
 \gamma_{4(n-1)} &\sim \Gamma_{8(nk+n-1)} : d^{8(n-2)(k+1)}, k(n-2)\bar{d}^4, (n-2)(k+2)\bar{\bar{d}}^4, \\ &\quad \gamma_{4(n-1)}^{8k+5}, [6nk+6n-8k-10]g, [6nk+6n-8k-10]h', I_{4k(n-2)}^4, \\
 \sigma_{nk+2n-1} &: d^{(n-2)(k+2)}, k(n-2)\bar{d}, 2(n-2)\bar{\bar{d}}, \gamma_{4(n-1)}^{k+1}, \\ &\quad [6nk+6n-8k-10]g, \\
 \sigma'_{3nk+4n-3} &: d^{(n-2)(3k+4)}, k(n-2)\bar{d}^2, (n-2)(k+2)\bar{\bar{d}}, 2(n-2)\bar{\bar{\bar{d}}}, \\ &\quad \gamma_{4(n-1)}^{3k+2}, [6nk+6n-8k-10]g, [6nk+6n-8k-10]h', I_{4k(n-2)}, \\
 J_{16(nk+n-1)} &: \Gamma_{8(nk+n-1)} + \tau_{2(nk+n-1)}^2 + \tau_{2(nk+n-1)}'^2
 \end{aligned}$$

This table gives the complete account of the composition of the transforms of the $(n-2)$ -fold line d , the base curve $\gamma_{4(n-1)}$ and of a general plane S_1 , as well as the multiplicities of all the fundamental elements on these transforms.

9. Intersections of the surfaces. A detailed account of the intersections of the surfaces connected with the transformation may be given as follows:

Two surfaces of the web, $S_{4nk+4n-3} = 0$ and $S'_{4nk+4n-3} = 0$, intersect in d to multiplicity $16(n-2)^2(k+1)^2$; $\gamma_{4(n-1)}$ to multiplicity $(4k+2)^2$; the $6nk+6n-8k-10$ lines (g); the $6nk+6n-8k-10$ lines (h'); d counted $4(n-2)(k+3) + 4k(n-2)$ times more due to contact along d ; $I_{4k(n-2)}$, counted four times; and a curve of order $4nk+4n-3$, image of the line common to the two planes of which the surfaces are the transforms.

An $S_{4nk+4n-3} = 0$ of the web and $\tau_{2(nk+n-1)} = 0$ intersect in d to multiplicity $8(n-2)^2(k+1)^2$; $\gamma_{4(n-1)}$ to multiplicity $2(2k+1)^2$; the $6nk+6n-8k-10$ lines (g); d counted $2k(n-2)$ times more due to contact along d ; $k(n-2)$ conics and the line d counted $4(n-2)$ times, the image of the point of intersection of the line d and the plane of which $S_{4nk+4n-3} = 0$ is the transform; and $I_{4k(n-2)}$, counted twice.

An $S_{4nk+4n-3} = 0$ of the web and $\tau'_{2(nk+n-1)} = 0$ intersect in d to multiplicity $8(n-2)^2(k+1)^2$; $\gamma_{4(n-1)}$ to multiplicity $2(2k+1)^2$; the $6nk+6n-8k-10$ lines (h'); d counted $2(n-2)(k+2)$ times more due to contact along d ; $k(n-2)$ conics, images of the point of intersection of d and the plane of which $S_{4nk+4n-3} = 0$ is the transform; and $I_{4k(n-2)}$, counted twice.

An $S_{4nk+4n-3} = 0$ of the web and $\sigma_{nk+n-1} = 0$ intersect in d to multiplicity $4(n-2)^2(k+1)(k+2)$; $\gamma_{4(n-1)}$ to multiplicity $(k+1)(4k+2)$; the $6nk+6n-8k-10$ lines (g); d counted $2k(n-2)$ times more due to contact along d ; and a curve of order $9nk+8n-12k-11$, invariant under (4).

An $S_{4nk+4n-3} = 0$ of the web and $\sigma'_{3nk+4n-3} = 0$ intersect

in d to multiplicity $4(n-2)^2(k+1)(3k+4)$; $\gamma_{4(n-1)}$ to multiplicity $(3k+2)(4k+2)$; the $6nk+6n-8k-10$ lines (g); the $6nk+6n-8k-10$ lines (h'); d counted $2(n-2)(k+2) + 4k(n-2)$ times more due to contact along d ; $I_{4k(n-2)}$, counted twice; and a curve of order $9nk+8n-12k-11$, invariant under (5).

An $S_{4nk+4n-3} = 0$ of the web and $\Gamma_{8(nk+n-1)} = 0$ intersect in d to multiplicity $32(n-2)^2(k+1)^2$; $\gamma_{4(n-1)}$ to multiplicity $(4k+2)(8k+3)$; the $6nk+6n-8k-10$ lines (g), counted twice; the $6nk+6n-8k-10$ lines (h'), counted twice; d counted $8(n-2)(k+2) + 8k(n-2)$ times more due to contact along d ; $4(n-1)$ curves of order $4k+2$, images of the points of intersection of d and the plane of which $S_{4nk+4n-3} = 0$ is the transform; and $I_{4k(n-2)}$, counted eight times.

The surfaces $\tau_{2(nk+n-1)} = 0$ and $\tau'_{2(nk+n-1)} = 0$ intersect in d to multiplicity $2(n-2)^2(k+1)^2$; $\gamma_{4(n-1)}$ to multiplicity $(2k+1)^2$; $I_{4k(n-2)}$; and $2(n-2)(k+1)$ conics (cf. Section 7).

The surfaces $\tau_{2(nk+n-1)} = 0$ and $\sigma_{nk+2n-1} = 0$ intersect in d to multiplicity $2(n-2)^2(k+2)$; $\gamma_{4(n-1)}$ to multiplicity $(2k+1)(k+1)$; the $6nk+6n-8k-10$ lines (g); and d counted $2k(n-2)$ times more due to contact along d .

The surfaces $\tau_{2(nk+n-1)} = 0$ and $\sigma'_{3nk+4n-3} = 0$ intersect in d to multiplicity $2(n-2)^2(k+1)(3k+4)$; $\gamma_{4(n-1)}$ to multiplicity $(2k+1)(3k+2)$; the $6nk+6n-8k-10$ lines (g); d counted $2k(n-2)$ times more due to contact along d ; $I_{4k(n-2)}$; and $2(n-2)(k+1)$ conics.

The surfaces $\tau_{2(nk+n-1)} = 0$ and $\Gamma_{8(nk+n-1)} = 0$ intersect in d to multiplicity $16(n-2)^2(k+1)^2$; $\gamma_{4(n-1)}$ to multiplicity $(2k+1)(8k+3)$; the $6nk+6n-8k-10$ lines (g), counted twice; d counted $4k(n-2)$ times more due to contact along d ; $4k(n-2)$ conics and the line d counted $8(n-2)$ times, images of the points of intersection

of the line d and $\gamma_{4(n-1)}$; and $I_{4k(n-2)}$, counted four times.

The surfaces $\tau_2'(nk+n-1) = 0$ and $\sigma_{nk+2n-1} = 0$ intersect in d to multiplicity $2(n-2)^2(k+1)(k+2)$; $\gamma_{4(n-1)}$ to multiplicity $(2k+1)(k+1)$; and a curve of order $8nk+6n-12k-10$ invariant under (4).

The surfaces $\tau_2'(nk+n-1) = 0$ and $\sigma_{3nk+4n-3}' = 0$ intersect in d to multiplicity $2(n-2)^2(k+1)(3k+4)$; $\gamma_{4(n-1)}$ to multiplicity $(2k+1)(3k+2)$; the $6nk+6n-8k-10$ lines (h'); d counted $2(n-2)(k+2)$ times more due to contact along d ; and $I_{4k(n-2)}$, counted twice.

The surfaces $\tau_2'(nk+n-1) = 0$ and $\Gamma_{8(nk+n-1)} = 0$ intersect in d to multiplicity $16(n-2)^2(k+1)^2$; $\gamma_{4(n-1)}$ to multiplicity $(2k+1)(8k+3)$; the $6nk+6n-8k-10$ lines (h'), counted twice; d counted $4(n-2)(k+2)$ times more due to contact along d ; $4k(n-2)$ conics, images of the points of intersection of d and $\gamma_{4(n-1)}$; and $I_{4k(n-2)}$, counted four times.

The surfaces $\sigma_{nk+2n-1} = 0$ and $\sigma_{3nk+4n-3}' = 0$ intersect in d to multiplicity $(n-2)^2(k+2)(3k+4)$; $\gamma_{4(n-1)}$ to multiplicity $(k+1)(3k+2)$; the $6nk+6n-8k-10$ lines (g); d counted $2k(n-2)+4(n-2)$ times more due to contact along d ; the $2k$ curves of order $3n-4$, $C_{2k(3n-4)}$; and C_{4n-3} .

The surfaces $\sigma_{nk+2n-1} = 0$ and $\Gamma_{8(nk+n-1)} = 0$ intersect in d to multiplicity $8(n-2)^2(k+1)(k+2)$; $\gamma_{4(n-1)}$ to multiplicity $(k+1)(8k+3)$; the $6nk+6n-8k-10$ lines (g), counted twice; d counted $4k(n-2)$ times more due to contact along d ; and a curve of order $20nk+16n-28k-24$, invariant under (4).

The surfaces $\sigma_{3nk+4n-3}' = 0$ and $\Gamma_{8(nk+n-1)} = 0$ intersect in d to multiplicity $8(n-2)^2(k+1)(3k+4)$; $\gamma_{4(n-1)}$ to multiplicity $(3k+2)(8k+3)$; the $6nk+6n-8k-10$ lines (g), counted twice; the $6nk+6n-8k-10$ lines (h'), counted twice; d counted $4(n-2)(k+2)$

+ $8k(n-2)$ times more due to contact along d ; $I_{4k(n-2)}$, counted four times; and a curve of order $24nk+16n-36k-24$, invariant under (5).

VITA

Frank Ayres, Jr., was born in Rock Hall, Maryland, on December 10, 1901. He received his secondary education in the public schools of Rock Hall and was granted the degree of Bachelor of Science by Washington College, Chestertown, Maryland, in 1921. He was in residence at the Johns Hopkins University during the summer session of 1923.

He entered the University of Chicago in the Summer Quarter of 1924 where he did graduate work under Professors Barnard, Bartky, Bliss, Dickson, Graves, Lane, Logsdon, Lunn, McMillan, Reid, Slaughter, Murnaghan, Blaschke, and Bompiani; and Doctors Bol, Hull, and Wilcox during the Summer, Autumn, Winter, and Spring Quarters of 1926-27, and the Summer Quarters of 1930, 1932, 1936, 1937, and 1938. He received the degree of Master of Science in 1927.

He wishes to acknowledge his indebtedness to Professor Logsdon for suggestions and criticism throughout the preparation of this dissertation.

